Demand & Comparative Statics

ECON 410

May 23, 2023

Introduction

- In the previous section, we discussed how to derive optimal consumption bundles
- Notice that we always derived optimal bundles given a *particular* set of prices and income
- In this section, we'll be more general and derive optimal bundles for *any* set of prices and income
- In other words, we'll derive x^* and y^* as functions of (p_x, p_y, I)

Demand

- The *demand function* for good x, denoted $x(p_x, p_y, I)$, gives the quantity of x consumed given income and prices
- In other words, given (p_x, p_y, I), the demand function simply reports the optimal amount of x to consume
- Mechanically, deriving a demand function is the same as deriving an optimal bundle
- But, demand functions are more general
  - Gives the optimal bundle for any set of (p_x, p_y, I), not just a particular price and income level

Deriving Demand (Example)

$$u(x, y) = x^{1/2}y^{1/2}$$
$$I = p_x x + p_y y$$

- Let's derive the demand functions for x and y
- First, set MRS=MRT:
$$\frac{y}{x} = \frac{p_x}{p_y}$$
- Solve for one of the variables (let's say y) using the equation above:
$$y = \frac{p_x}{p_y}x$$

Deriving Demand (Example)

- Plug the previous equation into the budget line:

$$I = p_x x + p_y y$$
$$I = p_x x + p_y \left(\frac{p_x}{p_y} x \right)$$

- Then, we can solve for x^*:

$$x^* = \frac{I}{2p_x}$$

- Plugging this into our equation for y yields y's demand function:

$$y^* = \frac{I}{2p_y}$$

Deriving Demand (Example)

$$x^* = \frac{I}{2p_x}$$
$$y^* = \frac{I}{2p_y}$$

- Having the optimal (x^*, y^*) expressed as functions of income and prices is useful
- We can use *comparative statics* to determine how consumption changes with respect to prices and income
- *Comparative statics* is just a convoluted way of saying "take derivatives"

Comparative Statics

Comparative Statics

$$x^* = f(I, p_x, p_y)$$
$$y^* = g(I, p_x, p_y)$$

- When we talk about comparative statics, really all we're talking about is differentiating demand with respect to model parameters
- 3 parameters of interest: I, p_x, and p_y
- Let's go through each one by one

Comparative Statics - Income

$$x^* = \frac{I}{2p_x} \qquad y^* = \frac{I}{2p_y}$$

- Back to our example. First, we'll talk about the income derivatives.
- Differentiating with respect to I yields:

$$\frac{\partial x}{\partial I} = \frac{1}{2p_x} > 0$$
$$\frac{\partial y}{\partial I} = \frac{1}{2p_y} > 0$$

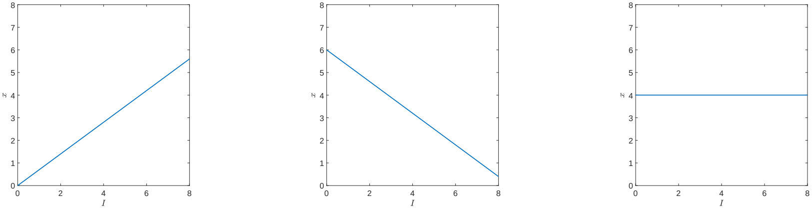
- Consumption of both x and y increases with income
- x and y are *normal goods*

Comparative Statics - Income Summary

$\frac{\partial x}{\partial I} > 0$	$\frac{\partial x}{\partial I} < 0$	$\frac{\partial x}{\partial I} = 0$
Normal	Inferior	Income Neutral

- Normal goods $\rightarrow$ consumption increases with income
  - Ex: Tito's vodka
- Inferior goods $\rightarrow$ consumption decreases with income
  - Ex: Burnette's vodka
- Income Neutral $\rightarrow$ consumption doesn't change with income
  - Ex: Insulin

Comparative Statics - Income Summary



(a) Normal (b) Inferior (c) Income Neutral

- The graphical depiction of x's demand as a function of income is referred to as the *Engel Curve*
- Named after 17th century economist **Ernst Engel**

Comparative Statics - Own Price

$$x^* = \frac{I}{2p_x} \quad , \quad y^* = \frac{I}{2p_y}$$

- Next for the own-price derivatives
- Differentiating x and y with respect to their prices yields:

$$\frac{\partial x}{\partial p_x} = -\frac{I}{2p_x^2} < 0$$
$$\frac{\partial y}{\partial p_y} = -\frac{I}{2p_y^2} < 0$$

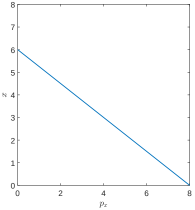
- Consumption of both x and y decreases in their prices
- x and y both follow the *law of demand* (LOD)

Comparative Statics - Own Price Summary

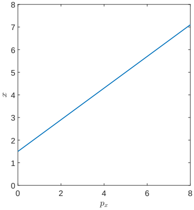
$\frac{\partial x}{\partial p_x} > 0$	$\frac{\partial x}{\partial p_x} < 0$	$\frac{\partial x}{\partial p_x} = 0$
Giffen	LOD	Price Neutral (Never happens)

- Consumption of x increases in $p_x \rightarrow$ Giffen good
  - Ex: ?
- Consumption of x decreases in $p_x \rightarrow$ LOD holds
  - Ex: most things

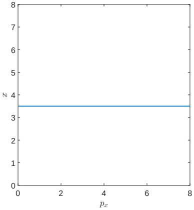
Comparative Statics - Own Price Summary



(d) Law of Demand



(e) Giffen



(f) Price Neutral

- If a good follows the law of demand, this simply means that its demand curve slopes down with respect to its price
- These are referred to as *demand curves*

Comparative Statics - Cross Price

$$x^* = \frac{I}{2p_x} \quad , \quad y^* = \frac{I}{2p_y}$$

- Lastly, the cross-price derivatives
- Differentiating x and y with respect to the other goods' price yields:

$$\frac{\partial x}{\partial p_y} = 0$$
$$\frac{\partial y}{\partial p_x} = 0$$

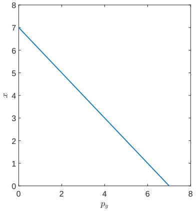
- Consumption of both x and y is unchanged by changes in the price of the other good
- x and y are *unrelated* goods

Comparative Statics - Cross Price Summary

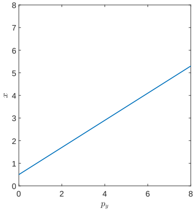
$\frac{\partial x}{\partial p_y} > 0$	$\frac{\partial x}{\partial p_y} < 0$	$\frac{\partial x}{\partial p_y} = 0$
Substitutes	Complements	Unrelated

- Substitutes: p_y increases $\rightarrow$ buy more of x
- Complements: p_y increases $\rightarrow$ buy less of x
- Unrelated: p_y increases $\rightarrow$ irrelevant for x

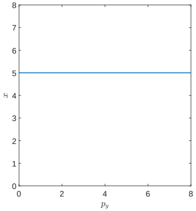
Comparative Statics - Cross Price Summary



(g) Complements



(h) Substitutes



(i) Unrelated

- Complements: good y gets more expensive $\rightarrow$ buy less of x since you like consuming x & y together
- Substitutes: good y gets more expensive $\rightarrow$ substitute away and buy more x

Elasticity

Elasticity

- One way to measure the responsiveness of a good's demand to changes in parameters is by taking derivatives
- Derivatives give the *unit* change in x as I, p_x, or p_y increase by one unit
- An alternative is to use elasticities
- Elasticities give the *percent* change in x as I, p_x, or p_y increase by one percent

Elasticity

- Why bother with elasticities?
- Consider the following scenario:
  - x costs \$5
  - If I get \$1 in additional income, I buy 10 more units of x
  - Then, $\frac{\partial x}{\partial I} = 10$
  - Seems like x is very sensitive to income
- Let's repeat the same scenario, but use Japanese Yen as the unit of income:
  - x costs ¥670 (\$1 $\approx$ ¥134)
  - If I get ¥1 in additional income, I buy .075 more units of x
  - Then, $\frac{\partial x}{\partial I} = .075$
  - Seems like x is not very sensitive to income

Elasticity

- Of course, changing the units in which we measure income doesn't change the nature of good x or its income sensitivity
- Point is: the units we use have a major impact on the values of price & income derivatives
- It would be nice to have a unit-free measure of change
- This is what we get with elasticities
- Rather than think about unit changes, we think about percent changes

Quick Note on Percent Change

- For a generic function $f(x)$, its percent change is given by:

$$\%\Delta f(x) = \frac{f'(x)}{f(x)}$$

- Notice that $\%\Delta f(x)$ is the same as $\frac{dlog(f(x))}{dx}$
- Differentiating $f(x)$ gives its unit change (wrt x), while differentiating $log(f(x))$ gives its % change (wrt x)

Elasticity

- Back to elasticities
- We'll consider three types of elasticities:
  1. Income elasticity
  2. Own-price elasticity
  3. Cross-price elasticity
- Let's start with the first type

Income Elasticity

- The income elasticity of good x, denoted ϵ_I, is given by:

$$\epsilon_I = \frac{\partial log(x)}{\partial log(I)} = \frac{\partial x}{x}\frac{I}{\partial I} = \frac{\partial x}{\partial I}\frac{I}{x}$$

- % change in x over % change in I
- Income elasticity is simply the income derivative times the ratio $\frac{I}{x}$
- Let's go through an example

Income Elasticity

- Recall the demand functions from our previous example:

$$x^* = \frac{I}{2p_x}$$
$$y^* = \frac{I}{2p_y}$$

- Let's derive the income elasticity of good x

- All we need to do:
  1. Compute $\frac{\partial x}{\partial I}$
  2. Multiply $\frac{\partial x}{\partial I}$ by $\frac{I}{x}$

Income Elasticity

- As we saw before, $\frac{\partial x}{\partial I}$ is given by:

$$\frac{\partial x}{\partial I} = \frac{1}{2p_x}$$

- Recalling the income-elasticity formula:

$$\begin{aligned} \epsilon_I &= \frac{\partial x}{\partial I}\frac{I}{x} \\ &= \frac{1}{2p_x}\frac{I}{x} \\ &= \frac{1}{2p_x}\frac{I}{\frac{I}{2p_x}} \\ &= 1 \end{aligned}$$

Income Elasticity

- We've determined that $\epsilon_I = 1$
- Interpretation: if I increases by 1%, x increases by 1% in response
- x has *unit* income elasticity
  - Note: Cobb-Douglas utility always yields unit income elasticities
- More generally:

$\|\epsilon_I\| > 1$	$\|\epsilon_I\| < 1$	$\|\epsilon_I\| = 1$
Elastic (wrt I)	Inelastic (wrt I)	Unit Elastic (wrt I)

- $|\epsilon_I|$ denotes the absolute value of ϵ_I

Own-Price Elasticity

$$x^* = \frac{I}{2p_x}$$

- Now, let's derive the price elasticity (or *own-price* elasticity) of x
- The price elasticity, denoted ϵ_x, is given by:

$$\epsilon_x = \frac{\partial log(x)}{\partial log(p_x)} = \frac{\partial x}{x}\frac{p_x}{\partial p_x} = \frac{\partial x}{\partial p_x}\frac{p_x}{x}$$

- Simply the price derivative times the ratio $\frac{p_x}{x}$

Own-Price Elasticity

- In this example, $\frac{\partial x}{\partial p_x}$ is given by:

$$\frac{\partial x}{\partial p_x} = -\frac{I}{2p_x^2}$$

- Plugging this into the price elasticity formula:

$$\begin{aligned}
\epsilon_x &= \frac{\partial x}{\partial p_x}\frac{p_x}{x} \\
&= -\frac{I}{2p_x^2}\frac{p_x}{\frac{I}{2p_x}} \\
&= -\frac{I}{2p_x^2}\frac{p_x^2}{I} \\
&= -1
\end{aligned}$$

Own-Price Elasticity

- We've determined that $\epsilon_x = -1$
- Interpretation: if p_x increases by 1%, x decreases by 1% in response
- x has *unit* price elasticity
  - Note: Cobb-Douglas utility always yields unit price elasticities
- More generally:

$\lvert\epsilon_x\rvert > 1$	$\lvert\epsilon_x\rvert < 1$	$\lvert\epsilon_x\rvert = 1$
Elastic (wrt p_x)	Inelastic (wrt p_x)	Unit Elastic (wrt p_x)

Cross-Price Elasticity

$$x^* = \frac{I}{2p_x}$$

- Finally, let's derive the cross-price elasticity of x
- The cross-price elasticity of good x, denoted ϵ_{xy}, is given by:

$$\epsilon_{xy} = \frac{\partial log(x)}{\partial log(p_y)} = \frac{\partial x}{x} \frac{p_y}{\partial p_y} = \frac{\partial x}{\partial p_y} \frac{p_y}{x}$$

- Simply the cross-price derivative times the ratio $\frac{p_y}{x}$
- Note: y's cross price elasticity is denoted ϵ_{yx}

Cross-Price Elasticity

- The cross-price derivative in this case is:

$$\frac{\partial x}{\partial p_y} = 0$$

- Plugging this into the cross-price elasticity formula gives:

$$\begin{aligned}
\epsilon_{xy} &= \frac{\partial x}{\partial p_y}\frac{p_y}{x} \\
&= 0 \times \frac{p_y}{x} \\
&= 0
\end{aligned}$$

Cross-Price Elasticity

- In this example, $\epsilon_{xy} = 0$
- Interpretation: if p_y increases by 1%, x doesn't change at all
  - Note: for Cobb-Douglas, x and y are always *unrelated* goods
- x is inelastic with respect to y's price
- More generally (same as before):

$\lvert\epsilon_{xy}\rvert > 1$	$\lvert\epsilon_{xy}\rvert < 1$	$\lvert\epsilon_{xy}\rvert = 1$
Elastic (wrt p_y)	Inelastic (wrt p_y)	Unit Elastic (wrt p_y)

Elasticity

- In summary: elasticities are a useful measure of how sensitive a good's demand is with respect to prices & income (or anything else)
- While derivatives measure unit change in response to a unit change, elasticities report the % change in response to a % change
- To compute elasticity for good x:
  - Derive the demand function for x
  - Take the relevant derivative
  - Plug the demand and its relative derivative into the elasticity formula

Consumption Curves

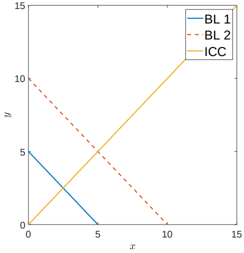
Consumption Curves

- Comparative statics are useful
  - Can determine how the demand of a good changes with respect to prices and income
- But, in a sense we can do even better
- Using the *income consumption curve* and *price consumption curve*, we can determine how the entire bundle (x^*, y^*) changes with respect to income and prices
- Two types of consumption curves:
  1. Income consumption curve (ICC)
  2. Price consumption curve (PCC)

Income Consumption Curve

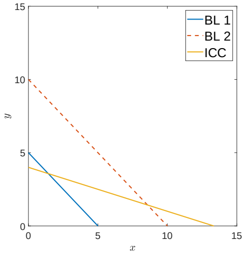
- The ICC describes how the optimal bundle (x^*, y^*) changes as income changes
- Each point on the ICC is an optimal bundle for some level of income
  - Traces out the optimal bundles as income changes
- Given a utility function $u(x, y)$, to plot its ICC, all we need is:
  - Sign of $\frac{\partial x}{\partial I}$
  - Sign of $\frac{\partial y}{\partial I}$
- In words: we need the demand functions for x and y along with the signs of their income derivatives
- 5 possible cases, let's go through each

Income Consumption Curve (Upward Sloping)



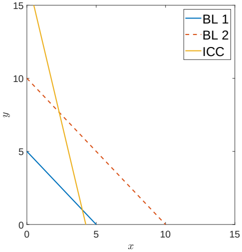
- Each point along the ICC is an optimal bundle (for some level of I)
- If both goods are normal ($\frac{\partial x}{\partial I} > 0$ and $\frac{\partial y}{\partial I} > 0$), the ICC slopes up
- As income increases (we move from BL 1 to BL 2), we consume more of both x and y

Income Consumption Curve (Downward Sloping #1)



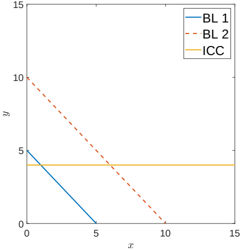
- If x is normal ($\frac{\partial x}{\partial I} > 0$) but y is inferior ($\frac{\partial y}{\partial I} < 0$), ICC slopes down
- As we get more income, we buy more x and less y

Income Consumption Curve (Downward Sloping #2)



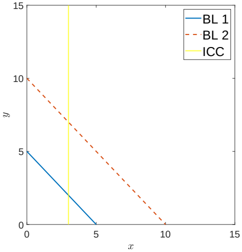
- If x is inferior ($\frac{\partial x}{\partial I} < 0$) but y is normal ($\frac{\partial y}{\partial I} > 0$), ICC again slopes down
- As we get more income, buy less x and more y

Income Consumption Curve (Horizontal)



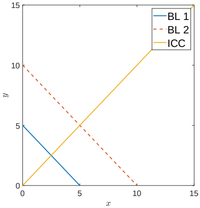
- If x is normal ($\frac{\partial x}{\partial I} > 0$) but y is income-neutral ($\frac{\partial y}{\partial I} = 0$), ICC is horizontal
- As income increases, we buy more x but the same amount of y

Income Consumption Curve (Vertical)

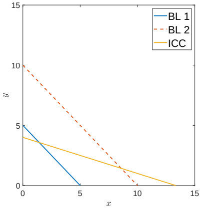


- Lastly, if x is income-neutral ($\frac{\partial x}{\partial I} = 0$) but y is normal ($\frac{\partial y}{\partial I} > 0$), ICC is vertical
- As income increases, we but more y but the same amount of x

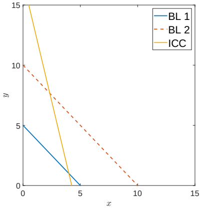
Income Consumption Curves (Summary)



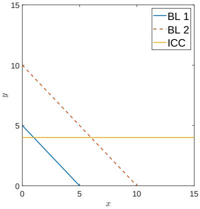
(j) $\frac{\partial x}{\partial I} > 0$ & $\frac{\partial y}{\partial I} > 0$



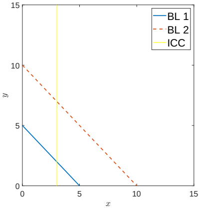
(k) $\frac{\partial x}{\partial I} > 0$ & $\frac{\partial y}{\partial I} < 0$



(l) $\frac{\partial x}{\partial I} < 0$ & $\frac{\partial y}{\partial I} > 0$



(m) $\frac{\partial x}{\partial I} > 0$ & $\frac{\partial y}{\partial I} = 0$

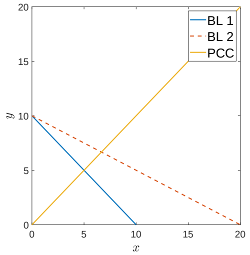


(n) $\frac{\partial x}{\partial I} = 0$ & $\frac{\partial y}{\partial I} > 0$

Price Consumption Curve

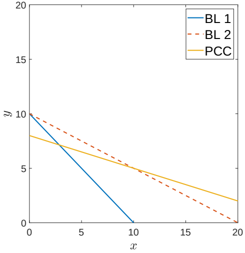
- Similarly, the PCC describes how the optimal bundle (x^*, y^*) changes as p_x changes
- Traces out the optimal bundles as p_x changes
- Note: we could do PCCs in terms of p_y if we wanted, it is just conventionally done in terms of p_x
- Given a utility function $u(x, y)$, to plot its PCC, all we need is:
  - Sign of $\frac{\partial x}{\partial p_x}$
  - Sign of $\frac{\partial y}{\partial p_x}$
- In words: we need the demand functions for x and y along with the signs of their p_x derivatives
- Again, there are 5 possible PCCs, let's go through each

Price Consumption Curve (Upward Sloping)



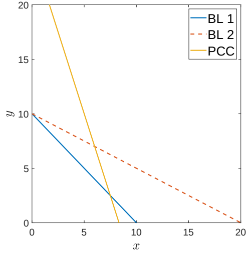
- PCC slopes up if:
  - x follows law of demand ($\frac{\partial x}{\partial p_x} < 0$)
  - x and y are complements ($\frac{\partial y}{\partial p_x} < 0$)
- p_x goes down $\rightarrow$ consume more of both x and y

Price Consumption Curve (Downward Sloping #1)



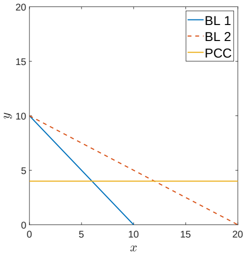
- PCC slopes (moderately) downwards if:
  - x follows law of demand ($\frac{\partial x}{\partial p_x} < 0$)
  - x and y are substitutes ($\frac{\partial y}{\partial p_x} > 0$)
- p_x goes down $\rightarrow$ consume more x and less y

Price Consumption Curve (Downward Sloping #2)



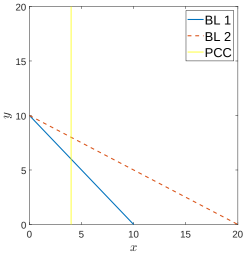
- PCC slopes (rapidly) downwards if:
  - x is a Giffen good ($\frac{\partial x}{\partial p_x} > 0$)
  - x and y are complements ($\frac{\partial y}{\partial p_x} < 0$)
- p_x goes down $\rightarrow$ consume less x and more y

Price Consumption Curve (Horizontal)



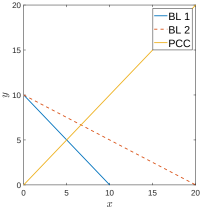
- PCC is horizontal if:
  - x follows the law of demand ($\frac{\partial x}{\partial p_x} < 0$)
  - x and y are unrelated ($\frac{\partial y}{\partial p_x} = 0$)
- p_x goes down $\rightarrow$ consume more x and the same amount of y

Price Consumption Curve (Vertical)

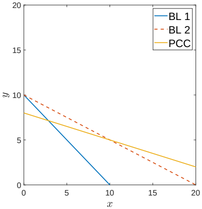


- PCC is vertical if:
  - x is "price-neutral" ($\frac{\partial x}{\partial p_x} = 0$)
  - x and y are complements ($\frac{\partial y}{\partial p_x} < 0$)
- p_x goes down $\rightarrow$ consume more y and the same amount of x
- This never happens

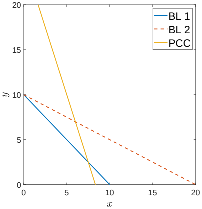
Price Consumption Curves (Summary)



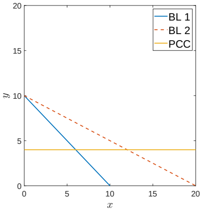
(o) $\frac{\partial x}{\partial p_x} < 0$ & $\frac{\partial y}{\partial p_x} < 0$



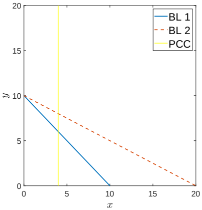
(p) $\frac{\partial x}{\partial p_x} < 0$ & $\frac{\partial y}{\partial p_x} > 0$



(q) $\frac{\partial x}{\partial p_x} > 0$ & $\frac{\partial y}{\partial p_x} < 0$



(r) $\frac{\partial x}{\partial p_x} < 0$ & $\frac{\partial y}{\partial p_x} = 0$



(s) $\frac{\partial x}{\partial p_x} = 0$ & $\frac{\partial y}{\partial p_x} < 0$

Indirect Utility

Indirect Utility

$$v(p_x, p_y, I) = \max_{x,y} \; u(x, y) = u(x^*, y^*)$$

- Given prices and income, p_x, p_y, I, the indirect utility function gives the highest possible level of utility
- Also known as the "value function"
- Given that the DM behaves optimally, the indirect utility function gives their payoff under any set of prices and income
- It is simply the "direct" utility function $u(x, y)$ evaluated at the demand functions x^* and y^*

Indirect Utility

$$v(p_x, p_y, I) = \max_{x,y} \; u(x, y) = u(x^*, y^*)$$

- With demand functions, we used comparative statics to assess how demand would change in response to changes in prices or income
- With the indirect utility function, we can assess how individual well-being changes in response to changes in prices or income
- For example: what's the marginal value of a dollar?
  - i.e. how would my utility increase if I had \$1 more?
- Let's find a very general answer to this question

Indirect Utility

- First, there is an important connection between the value function and Lagrangian that we should point out
- Recall the general form of the Lagrangian:

$$\mathscr{L}(x, y, \lambda, p_x, p_y, I) = u(x, y) + \lambda(I - p_x x - p_y y)$$

- Truly, the Lagrangian is a function of the variables (x, y, λ) and parameters (p_x, p_y, I)
- Let's make quick note of the fact that:

$$\frac{\partial \mathscr{L}}{\partial I} = \lambda$$

Indirect Utility

- What about the Lagrangian evaluated at the optimal (x^*, y^*)?

$$\mathscr{L}(x^*, y^*, \lambda, p_x, p_y, I) = u(x^*, y^*) + \lambda(I - p_x x^* - p_y y^*)$$

- Remember that at the optimal bundle, we exhaust all of our income:

$$I = p_x x^* + p_y y^*$$

- The second term in the Lagrangian is thus $= 0$

- Then we have:

$$\mathscr{L}(x^*, y^*, \lambda, p_x, p_y, I) = u(x^*, y^*) = v(p_x, p_y, I)$$

Indirect Utility

$$\mathscr{L}(x^*, y^*, \lambda, p_x, p_y, I) = v(p_x, p_y, I)$$

- The Lagrangian evaluated at (x^*, y^*) is just equal to the value function
- Back to our question: what's the marginal value of a dollar?
- Answering this question is just a matter of differentiating the value function $v(p_x, p_y, I)$:

$$\frac{\partial v}{\partial I} = \frac{\partial \mathscr{L}}{\partial I} = \lambda$$

Envelope Theorem

$$\frac{\partial v}{\partial I} = \frac{\partial \mathscr{L}}{\partial I} = \lambda$$

- This is an important result in economics referred to as the *envelope theorem*
- It gives a concrete interpretation of the Lagrange multiplier λ
- λ measures the marginal utility of income, or put differently: the value of an additional dollar
    - If I got 1 more unit of income, my utility would increase by λ (given optimal behavior)

Indirect Utility

$$v(p_x, p_y, I) = \max_{x,y} \; u(x, y) = u(x^*, y^*)$$

- Back to the indirect utility function
- How would one derive the indirect utility function?
- Answer: derive the demand functions x^* and y^*, and just plug them into $u(x, y)$
- Let's do this using our example from the beginning of the slides

Indirect Utility

$$u(x, y) = x^{1/2}y^{1/2}$$

- Given the utility function above, we determined the its demand functions were given by:

$$x^* = \frac{I}{2p_x}$$
$$y^* = \frac{I}{2p_y}$$

- The indirect utility function $v(I, p_x, p_y) = u(x^*, y^*)$
- To derive indirect utility, simply plug x^* and y^* into the utility function

Indirect Utility

- Plugging the demand functions into $u(x, y)$ gives:

$$\begin{aligned} u(x^*, y^*) = v(I, p_x, p_y) &= \left(\frac{I}{2p_x}\right)^{1/2}\left(\frac{I}{2p_y}\right)^{1/2} \\ &= \frac{I}{2(p_x p_y)^{1/2}} \end{aligned}$$

- With indirect utility, we can study how utility changes with parameters (p_x, p_y, I) as opposed to choice variables (x, y)
- Allows us to assess how changes in market conditions (here just prices and income) impact individual welfare

Indirect Utility

$$v(I, p_x, p_y) = \frac{I}{2(p_x p_y)^{1/2}}$$

- Here, $\frac{\partial v}{\partial I} > 0$, implying the optimal level of utility increases with income
  - The more money I have, the more utility I can obtain
- $\frac{\partial v}{\partial p_x} < 0$ and $\frac{\partial v}{\partial p_y} < 0$, implying the optimal level of utility decreases with prices
  - The more expensive things are, the less utility I can obtain
- In fact, no matter the utility function, both of these things always hold
  - Optimal utility always increases with income
  - Optimal utility always decreases with prices