Input Demand

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Introduction

- Given a production function, $f(L, K)$, prices (w, r), and a production quota $q \leq f(L, K)$, we've talked about deriving the cost-minimizing production plan (L^*, K^*)
- Works basically the same as in the utility maximization section
- Next we'll talk through deriving the more general *input demand functions*
- Input demand functions reveal a lot about firms' costs, returns to scale, etc.

Input Demand Functions

$$L^* = g(q, w, r)$$
$$K^* = h(q, w, r)$$

- The input demand functions express the optimal L and K as functions of:
  - The production level q
  - Wage rate w
  - Capital rental rate r
- Deriving them works much the same as deriving any other demand function
- Let's go through an example

Inputs Demand (Example)

$$q = L^{1/2}K^{1/2}$$

- Suppose we're given the production function above and want to derive demand for L^* and K^*
- Begin by setting $MRTS = MRT$:

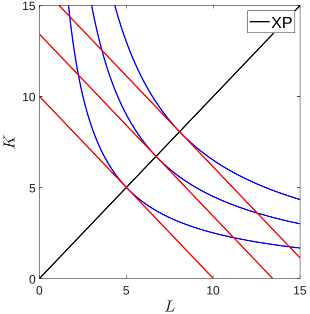
$$\frac{MP_L}{MP_K} = \frac{w}{r}$$
$$\frac{L^{-1/2}K^{1/2}}{L^{1/2}K^{-1/2}} = \frac{w}{r}$$
$$\frac{K}{L} = \frac{w}{r}$$

Expansion Path

$$\frac{K}{L} = \frac{w}{r}$$
$$K = \frac{w}{r}L$$

- As usual, we use the tangency condition to solve for one of the variables
- However, let's pause for a moment and talk about the $K = \frac{w}{r}L$ expression
- This is referred to as the capital *expansion path* (XP)
- Describes how the optimal production bundle changes as quantity increases

Expansion Path



- Expansion path traces out the optimal production plan as quantity increases
- A lot like the ICC
- Expansion path typically slopes up

Inputs Demand (Example)

$$K = \frac{w}{r} L$$

- Back to solving for input demand
- Take the expansion path, and plug into the production function:

$$q = K^{1/2} L^{1/2}$$

$$q = \left(\frac{w}{r}\right)^{1/2} L^{1/2} L^{1/2}$$

$$L^* = q \left(\frac{r}{w}\right)^{1/2}$$

- Above is the labor demand function
- To get capital's demand function, plug L^* into the expansion path

Inputs Demand (Example)

$$K = \frac{w}{r} L$$

$$K^* = \frac{w}{r} \left(\frac{r}{w} \right)^{1/2} q$$

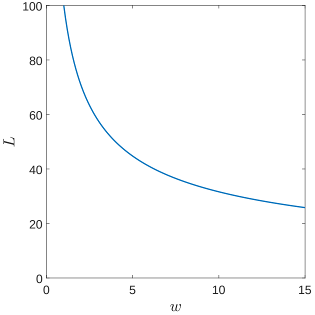
$$K^* = \left(\frac{w}{r} \right)^{1/2} q$$

- In summary, using the tangency condition and production function, we obtain the following input demand functions:

$$K^* = \left(\frac{w}{r} \right)^{1/2} q$$

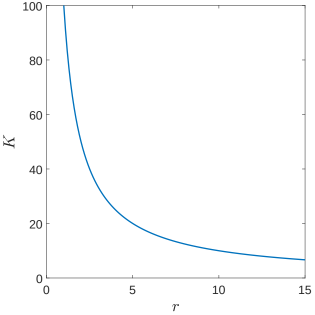
$$L^* = \left(\frac{r}{w} \right)^{1/2} q$$

Labor Demand



- The labor demand function gives the optimal number of labor units given quantity q and prices (w, r)
- Generally, labor demand decreases with respect to its price: $\frac{\partial L}{\partial w} < 0$

Capital Demand



- Similarly, the capital demand function gives the optimal number of capital units given quantity q and price (w, r)
- Generally, capital demand decreases with respect to its price: $\frac{\partial K}{\partial r} < 0$

Cost Functions

$$K^* = \left(\frac{w}{r}\right)^{1/2} q$$

$$L^* = \left(\frac{r}{w}\right)^{1/2} q$$

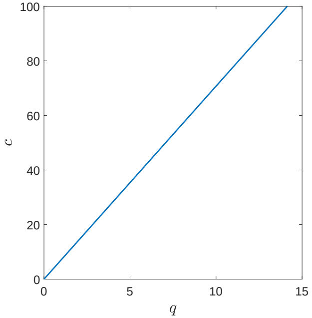
- Given the demand functions above, we can derive the firm's cost function $c(q)$
- Plug the demand functions into the isocost line:

$$c = wL + rK$$

$$c = w\left(\frac{r}{w}\right)^{1/2} q + r\left(\frac{w}{r}\right)^{1/2} q$$

$$c(q) = 2(wr)^{1/2} q$$

Cost Functions



- Cost function gives the cost of producing quantity q **given optimal firm behavior**
  - Minimum cost to produce q

Cost Functions

$$c(q) = 2(wr)^{1/2}q$$

- Let's quickly dissect the cost function and define a few objects
- $c(q)$ gives the *total cost* of production
- $\frac{\partial c}{\partial q}$ gives the *marginal cost*, $MC(q)$, of production
  - i.e. how much will it cost to produce 1 additional unit of q?
- $\frac{c(q)}{q}$ gives the *average cost*, $ATC(q)$, of production
  - i.e. how much is the firm spending per unit of output q?

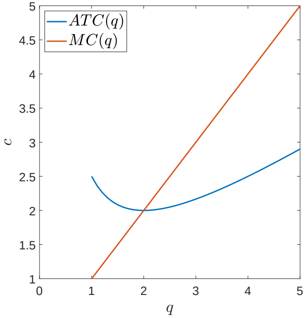
Average Costs & Marginal Costs

- Quick question: How do average costs change with production?
- Taking the derivative of $ATC(q)$:

$$\begin{aligned} ATC'(q) &= \frac{qc'(q) - c(q)}{q^2} \\ &= \frac{MC(q)}{q} - \frac{ATC(q)}{q} \end{aligned}$$

- If $MC(q) > ATC(q)$, then average costs increase with q
  - Next unit costs more than average to produce, average comes up
- If $MC(q) < ATC(q)$, then average costs decrease with q
  - Next unit costs less than average to produce, average comes down

Average Costs & Marginal Costs



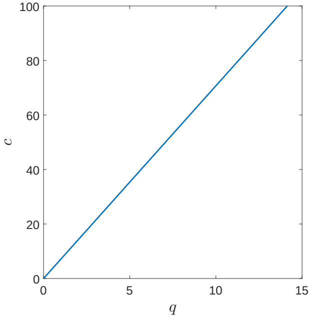
- Average costs fall if $MC(q) < ATC(q)$
- Average costs rise if $MC(q) > ATC(q)$
- Notice $ATC(q)$ is minimized when $ATC(q) = MC(q)$

Fixed versus Variable Costs

$$c(q) = FC + VC(q)$$

- The cost functions we'll see are of the form above
- Firm's *fixed costs*, FC, are the costs they incur no matter how much they produce
  - Ex: rent for their office building
- Firm's *variable costs*, $VC(q)$, are the costs which vary with production
  - Ex: firm must buy more shipping materials as production increases
- Average fixed costs (AFC) are given by: $\frac{FC}{q}$
- Average variable costs (AVC) are given by: $\frac{VC(q)}{q}$

Cost Functions



- Back to our example. The production function was $q = L^{1/2}K^{1/2}$
  - This exhibits constant returns to scale
- The resulting cost function was linear
- This was no coincidence

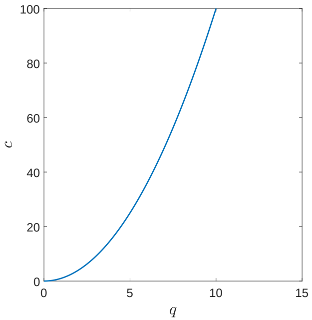
Returns to Scale & Firm Costs

- Let's recall the three cases of returns to scale
- *Decreasing Returns to Scale*: increasing inputs by $t > 1$ increases output by less than t
- *Increasing Returns to Scale*: increasing inputs by $t > 1$ increases output by more than t
- *Constant Returns to Scale*: increasing inputs by $t > 1$ increases output by t
- Which of the three cases we're in has implications for the marginal costs of production

Returns to Scale & Firm Costs

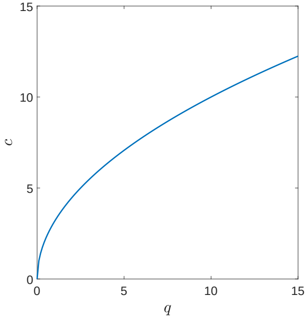
- If a production function exhibits DRS, the resulting cost function will be convex in q
  - Marginal cost of production increases
  - Eventually becomes prohibitively expensive to increase quantity
- If a production function exhibits IRS, the resulting cost function will be concave in q
  - Marginal cost of production decreases
  - Gradually becomes cheaper and cheaper to produce additional output
- If a production function exhibits CRS, the resulting cost function will be linear in q
  - Marginal cost of production is constant
  - Cost of increasing output is always the same

Cost Function (DRS)



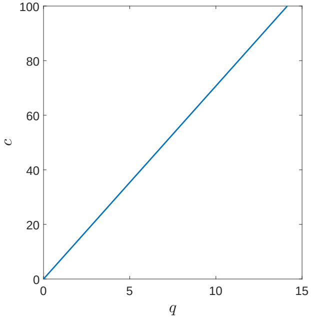
- If $q = f(L, K)$ exhibits DRS, marginal costs increase $(c''(q) > 0)$
- As a result, $c(q)$ is convex

Cost Function (IRS)



- If $q = f(L, K)$ exhibits IRS, marginal costs increase $(c''(q) < 0)$
- As a result, $c(q)$ is concave

Cost Function (CRS)



- If $q = f(L, K)$ exhibits CRS, marginal costs increase $(c''(q) = 0)$
- As a result, $c(q)$ is linear

Perfect Substitutes & Perfect Complements

- Let's quickly talk through the special cases of perfect substitutes and perfect complements
- Recall the general perfect substitutes production function:

$$q = aL + bK$$

- And the general perfect complements production function:

$$q = \min\{\frac{L}{a}, \frac{K}{b}\}$$

- We'll talk about the cost functions and expansion paths in these two cases

Perfect Substitutes

$$q = aL + bK$$

- What does the cost function look like for perfect substitutes?
- Optimal bundle will always be $(\frac{q}{a}, 0)$ or $(0, \frac{q}{b})$
- Then, the cost function is either:

$$c(q) = \frac{w}{a}q \quad \text{or} \quad c(q) = \frac{r}{b}q$$

- Cost function is always linear in q
  - Perfect substitutes production functions always exhibit CRS

Perfect Substitutes

$$q = aL + bK$$

- How about the expansion path for perfect substitutes?
- Again, we always either use all capital or all labor
- Then, the expansion path is either the vertical or horizontal axis on the LK plane
  - Vertical axis if all capital
  - Horizontal axis is all labor

Perfect Complements

$$q = \min\{\frac{L}{a}, \frac{K}{b}\}$$

- What does the cost function look like for perfect complements?
- Input demand functions are always given by:

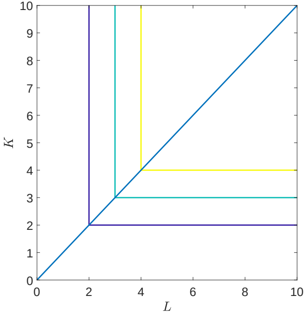
$$L^* = aq$$
$$K^* = bq$$

- Plugging these into the isocost line:

$$c = wL + rK$$
$$c = awq + brq = q(aw + br)$$

- Cost is again linear in q (CRS)

Perfect Complements



- How about the expansion path?
- Optimal production bundles always lie on the kink points of the isoquants
- Expansion path traces out the kink points

Summary

- Deriving input demand functions works basically the same as deriving consumer demand
- With convex isoquants, we solve for L^* & K^* using the two conditions:
  - $MRTS = MRT$
  - $q = f(L, K)$
- Plugging L^* & K^* into the isocost equation yields the cost function
  - Gives minimum cost to produce any level of q
- Curvature of the cost function is closely related to firms' returns to scale