Cost Minimization

Noah Lyman

June 5, 2023

Firm Profits

- Typically, we model firms as profit maximizers
- Firm profits, denoted by π, are given by:

$$\pi = pq - c(q)$$

- p is the price of the good
  - Firm gets p per unit of q
- $c(q)$ is the firm's *cost function*
  - Specifies how much it costs to produce q units of the good
- Before talking about profit maximization, we'll talk a bit about this cost function

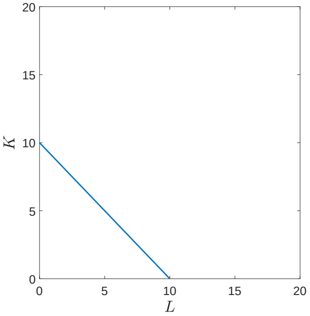
Cost Functions

- Resources (here K and L) cost money, so it is in the firm's interest to optimize their resource usage in such a way that minimizes costs
- In general, firms' total expenses c are given by:

$$c = wL + rK$$

- The above equation defines the *isocost* line
- w is the cost of each unit of labor (i.e. the wage rate)
- r is the cost of each unit of capital
- The goal is to find the optimal combination of L and K such that firms are incur the lowest cost possible when producing q

Isocost Line



- The isocost line contains all (L, K) combinations which cost c in total
- Mechanically, they work the same as budget lines

Slope of Isocost Line

$$c = wL + rK$$

- To derive the slope of the isocost line, first solve for K:

$$K = \frac{c}{r} - \frac{w}{r}L$$

- Then, differentiate with respect to L:

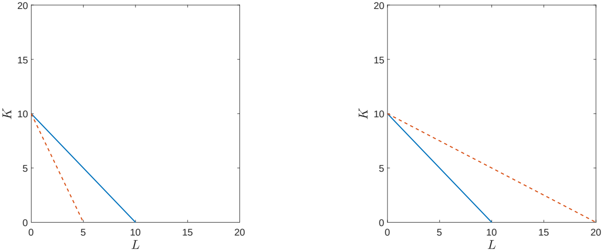
$$slope(isocost) = \frac{\partial K}{\partial L} = -\frac{w}{r} = -MRT$$

- The price ratio $\frac{w}{r}$ is again referred to as the *marginal rate of transformation* or MRT

Intercepts of the Isocost Line

- The L intercept is given by $\frac{c}{w}$
- Notice that as:
  - w increases, L intercept decreases
  - c increases, L intercept increases
  - r increases, L intercept is unchanged
- The K intercept is given by $\frac{c}{r}$
- Notice that as:
  - r increases, K intercept decreases
  - c increases, K intercept increases
  - w increases, K intercept is unchanged

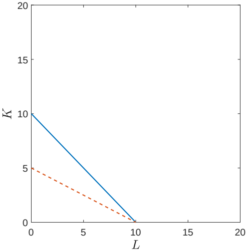
Isocost Lines - Comparative Statics



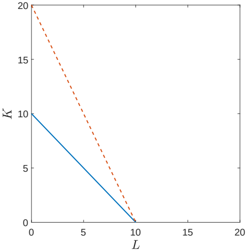
(a) $w \uparrow$ (b) $w \downarrow$

- If w increases, all else equal, the BL rotates inwards
- If w decreases, all else equal, the BL rotates outwards
- No change in K intercept

Isocost Lines - Comparative Statics



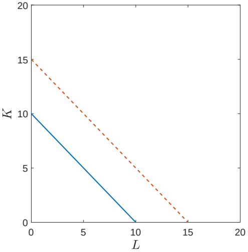
(c) $r \uparrow$



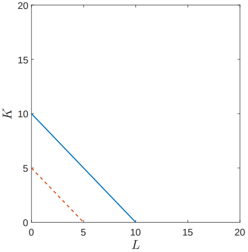
(d) $r \downarrow$

- If r increases, all else equal, the isocost rotates inwards
- If r decreases, all else equal, the isocost rotates outwards
- No change in L intercept

Isocost Lines - Comparative Statics



(e) $c \uparrow$



(f) $c \downarrow$

- If c increases, all else equal, the isocost shifts outwards
- If c decreases, all else equal, the isocost shifts inwards
- No change in slope

Cost Functions

- How are cost functions and isocost lines related?
- Total costs are always given by:

$$c = wL + rK$$

- The cost function $c(q)$ gives the cost of producing q units **given that the firm behaves optimally**:

$$c(q) = wL^*(q) + rK^*(q)$$

- $L^*(q)$ and $K^*(q)$ are the optimal quantities of L and K given that firm produces q
  - *Input demand functions*

Cost Functions

$$c(q) = wL^*(q) + rK^*(q)$$

- Isocost line evaluated at the optimal (L^*, K^*) gives the cost function
  - Very similar to the indirect utility function
- Behaves optimally $\rightarrow$ chooses the (L^*, K^*) combination which makes costs as low as possible
- How do we find (L^*, K^*)? This will be the focus of this chapter
- The approach we'll use will be very similar to what we did in the utility maximization
  - There is an important difference in approach that I'll point out
  - However, we'll see that both approaches are equivalent

Cost Minimization

- Let's begin by setting up a standard cost minimization problem
- Firm costs are again given by:

$$c = wL + rK$$

- Let's say they have a production quota, so need to produce at least q units:

$$q \leq f(L, K)$$

- The objective is to minimize c subject to the production quota

Cost Minimization

- One way to solve this problem is by using the Lagrangian approach
- The part we are optimizing is total costs, and our constraint is the production quota
- The Lagrangian in this case is:

$$\mathscr{L} = wL + rK + \lambda(q - f(L, K))$$

- Looks almost like the reverse of what we saw in the utility maximization chapter

Cost Minimization

$$\mathscr{L} = wL + rK + \lambda(q - f(L, K))$$

- Taking first order conditions:

$$\frac{\partial \mathscr{L}}{\partial L} = 0 \iff w - \lambda MP_L = 0$$
$$\frac{\partial \mathscr{L}}{\partial K} = 0 \iff r - \lambda MP_K = 0$$
$$\frac{\partial \mathscr{L}}{\partial \lambda} = 0 \iff q - f(L, K) = 0$$

- Like before, we want to eliminate λ by dividing the first condition by the second (after moving the λ terms to the RHS)

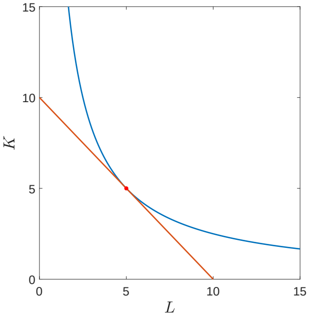
Cost Minimization

- Simplifying the system yields:

$$\frac{MP_L}{MP_K} = \frac{w}{r}$$
$$q = f(L, K)$$

- The first condition states that at the optimal (L^*, K^*), $MRTS = MRT$
- At the optimal production bundle, slope of the isoquant equals slope of the isocost line
  - (L^*, K^*) is the tangency point between the two curves
- The second simply spits back the production function

Tangency Condition



- Given convex isoquants, cost-minimizing production plan (L^*, K^*) lies at the isoquant/isocost tangency point

Tangency Condition

- If we want a more intuitive interpretation of the optimality condition, we can rearrange it slightly:

$$\frac{MP_L}{w} = \frac{MP_K}{r}$$

- LHS is the bang per buck for L
  - i.e. How much additional quantity per dollar spent on L?
- RHS is the bang per buck for K
  - i.e. How much additional quantity per dollar spent on K?
- At the optimum, the two BPBs are equal
  - Otherwise, we could swap some L for K (or vice versa) and reduce costs

Cost Minimization

- In summary: the optimal production bundle can be derived using the two conditions:

$$MRTS = MRT$$
$$q = f(L, K)$$

- 2 equations, 2 unknowns $\rightarrow$ solve for (L, K)
- Important note: the previous statement is true as long as isoquants are convex
  - If isoquants are not convex, we need to look out for corner solutions
- Let's work through an example

Cost Minimization (Example)

$$q = L^{1/2}K^{1/2}$$

- Suppose we're given the production function above and:
  - $w = 20,\ r = 5,\ q = 100$
- This production function is Cobb-Douglas, so we know it has convex isoquants
- Setting $MRTS = MRT$:

$$\frac{MP_L}{MP_K} = \frac{w}{r}$$

$$\frac{\frac{1}{2}L^{-1/2}K^{1/2}}{\frac{1}{2}L^{1/2}K^{-1/2}} = \frac{20}{5}$$

$$\frac{K}{L} = 4$$

$$K = 4L$$

Cost Minimization (Example)

$$K = 4L$$

- Plug the above condition into the production function:

$$q = L^{1/2}K^{1/2}$$
$$100 = L^{1/2}(4L)^{1/2}$$
$$100 = 2L$$
$$L^* = 50$$

- If $L^* = 50$, then $K^* = 200$ by the first condition on the slide
- Optimal bundle is thus $(L^*, K^*) = (50, 200)$

Cost Minimization (Example)

$$q = L^{1/2}K^{1/2}$$

- Given the production function above, along with prices $(w, r) = (20, 5)$ and the production quota $q = 100$, we found the optimal bundle is $(L^*, K^*) = (50, 400)$
- If the firm needs to produce $q = 100$, the cheapest way to do so is by using $L^* = 50$ and $K^* = 200$
- How much does this actually cost?

Cost Minimization (Example)

$$c = wL + rK$$

- To compute total costs, we just plug (L^*, K^*) along with their prices into the isocost line
- Doing so yields:

$$c = 20(50) + 5(200) = 1000 + 1000 = 2000$$

- If the firm needs to produce $q = 100$, the cheapest possible production plan costs $c = 2000$

Duality

- Let me quickly digress, and ask: how is what we just did similar/different do what we did with utility maximization?
- Recall the generic utility maximization problem:

$$\max_{x,y} \; u(x, y) \quad \text{subject to :}$$
$$I \geq p_x x + p_y y$$

- We maximize utility subject to not spending more than we have

Duality

$$\max_{x,y} \; u(x,y) \quad \text{subject to :}$$
$$I \geq p_x x + p_y y$$

- Take budget as given, maximize utility
- If we solve the problem above, what we obtain is the indirect utility function:
$$v(I, p_x, p_y) = \max_{x,y} u(x,y)$$
- $v(I, p_x, p_y)$ tells us how much utility we can obtain given income and prices

Duality

- An equivalent way to formulate the previous problem is:

$$\min_{x,y} \; p_x x + p_y y \quad \text{subject to :}$$
$$\bar{u} \leq u(x, y)$$

- Alternatively, we could minimize expenditures subject to attain some target level of utility $\bar{u}$
- Take utility as given, minimize expenditures
- This formulation and the previous one are equivalent
    - This minimization problem is the *dual* of the previous maximization problem

Duality

$$\min_{x,y} \; p_x x + p_y y \quad \text{subject to :}$$
$$\bar{u} \leq u(x, y)$$

- If we solve the problem above, we obtain what's called the *expenditure function*:

$$e(\bar{u}, p_x, p_y) = \min_{x,y} \; p_x x + p_y y = p_x x^* + p_y y^*$$

- The expenditure function tells us: given prices, how much will it cost to obtain utility $\bar{u}$?
- Sort of the reverse of the formulation we're familiar with

Duality

- The second formulation is exactly what we're doing when it comes to cost minimization
- We minimize costs subject to meeting some production quota
- More formally:

$$\min_{L,K} \quad wL + rK \quad \text{subject to :}$$
$$q \leq f(L, K)$$

- What we'll obtain is the cost function:

$$c(q, w, r) = \min_{L,K} wL + rK = wL^* + rK^*$$

Special Production Functions

- So far, we've focused on cost minimization in the case of convex isoquants
- In this case, we use the following two conditions to solve for L^* and K^*:

$$MRTS = MRT$$
$$q = f(L, K)$$

- Let's talk about two special cases where the approach above doesn't work:
  1. Perfect substitutes
  2. Perfect complements (i.e. fixed proportion)

Perfect Substitutes

$$q = aL + bK$$

- Perfect substitute production functions are linear in both L and K
- As a result, isoquants will also be linear (not strictly convex)
- Since the isoquants are not strictly convex, we know we'll have a corner solution
- Given that the firm must produce q units, there are two possible corner solutions:
  - Only labor: $(L^*, K^*) = (\frac{q}{a}, 0)$
  - Only capital: $(L^*, K^*) = (0, \frac{q}{b})$

Perfect Substitutes

- How do we determine which of the two corners is optimal?
- The optimal solution is the one which yields lower costs
- One way to determine which bundle is cheaper is by comparing bang per bucks:

$$\frac{MP_L}{w} \quad \text{vs} \quad \frac{MP_K}{r}$$

- Let's go through an example

Perfect Substitutes (Example)

$$q = 2L + K$$

- Suppose we're given the production function above, along with:
  - $w = 10,\ r = 2,\ q = 100$
- Perfect substitutes $\rightarrow$ corner solution
- To pick between the two corners, compare bang per bucks:

$$\frac{MP_L}{w} \quad \text{vs} \quad \frac{MP_K}{r}$$
$$\frac{2}{10} < \frac{1}{2}$$

- Optimal here to use only capital

Perfect Substitutes (Example)

$$q = 2L + K$$

- How much capital do we need to use?
- In this example, we needed $q = 100$. Plug this into the production function, set $L = 0$, and solve for K:

$$K^* = 100$$

- The optimal production bundle is thus $(L^*, K^*) = (0, 100)$

Perfect Complements

$$q = \min\{\frac{L}{a}, \frac{K}{b}\}$$

- Perfect complements production functions are not differentiable, so we'll need an alternative way to derive optimal production bundles
- As we saw in consumer theory, we know that at the optimum, the two things in the min will be equal
- We'll use this insight to derive optimal production bundles given a perfect complements production function
- Let's go through an example

Perfect Complements

$$q = \min\{\frac{L}{2}, \frac{K}{4}\}$$

- Suppose we're given the production function above, along with:
  - $w = 10$, $r = 2$, $q = 100$
- Set equal the things in the min:

$$\frac{L}{2} = \frac{K}{4}$$

- Since $q = \min\{\frac{L}{2}, \frac{K}{4}\}$, if $\frac{L}{2} = \frac{K}{4}$, then $q = \frac{L}{2}$ and $q = \frac{K}{4}$

Perfect Complements

- Since $q = \frac{L}{2}$ and $q = \frac{K}{4}$, use the fact that $q = 100$ to solve for L^* and K^*:

$$L^* = 200$$
$$K^* = 400$$

- The optimal production bundle is thus $(L^*, K^*) = (200, 400)$

Summary

- To derive optimal production bundles:
  - Set $MRTS = MRT$
  - Use tangency condition & $q = f(L, K)$ to derive (L^*, K^*)
- Given a level of production q, (L^*, K^*) generates the lowest possible cost of producing q
- The cost minimization setup is the "fraternal twin" of the utility maximization setup
  - Look different, but deep down are the same
- Only anticipate the following production functions moving forward:
  - Those with convex isoquants
  - Perfect substitutes
  - Perfect complements

Examples

- Convex Isoquants: $q(L, K) = \sqrt{L} + \sqrt{K}$
  - $q = 30,\ w = 1,\ r = 2$
- Perfect Substitutes: $q(L, K) = 4L + K$
  - $q = 100,\ w = 10,\ r = 2$
- Perfect Complements: $q(L, K) = \min\{L, \frac{K}{4}\}$
  - $q = 100,\ w = 10,\ r = 2$