# Production

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# Introduction

- We've spent the first half of the class discussing optimal decision making from the perspective of an individual
- Now, we'll shift perspectives and talk about optimal decision making for firms
- Individuals are consumers, while firms are producers
- For a given transaction, firms and consumers lie on opposite ends. Markets exist so these transactions can happen.
- There are deep similarities between consumer theory and firm theory

# Production Functions

- What are firms? Firms are organizations which convert inputs into outputs.
- We denote by $q$ the level of a given firm's output
- Typically, we assume production depends on two inputs:
  - Capital ($K$)
  - Labor ($L$)
- The *production function* specifies the relationship between $q$ and its inputs $(K, L)$:

$$q = f(K, L)$$

- $q$ is the quantity produced given $K$ units of capital and $L$ units of labor

# Production Functions

$$q = f(K, L)$$

- Production functions, like utility functions, are just functions
  - Specify output for a given combination of inputs
- Remember that utility functions had no cardinal interpretation
  - The value of $u(x, y)$ was meaningless, only the ordering had meaning
- Production functions on the other hand *do* have a cardinal interpretation
- The value of $q$ actually means something here
  - It is the amount that the firm produces, or the "supply"

# Production Functions

$$q = f(K, L)$$

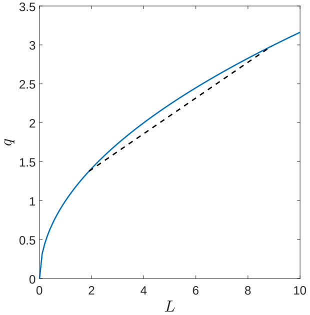
- Graphing production functions works basically the same as graphing utility functions
- To plot how $q$ changes with either $K$ or $L$, we'll need to check the signs of two derivatives
  - For $L$: $\frac{\partial f}{\partial L}$ and $\frac{\partial^2 f}{\partial L^2}$
  - For $K$: $\frac{\partial f}{\partial K}$ and $\frac{\partial^2 f}{\partial K^2}$
- Let's make sense of these derivatives before talking more about graphing

# Marginal Product of Labor

$$MP_L = \frac{\partial f}{\partial L}$$

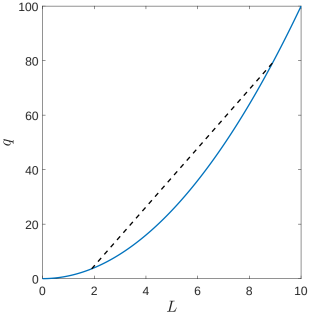
- The *marginal product of labor*, denoted by $MP_L$, is the change in production $q$ induced by adding one unit of labor
- We typically assume $MP_L \geq 0$
- How does $MP_L$ vary with $L$?
- There are three cases to consider:
  1. Diminishing $MP_L$
  2. Increasing $MP_L$
  3. Constant $MP_L$

# Diminishing Marginal Product of Labor



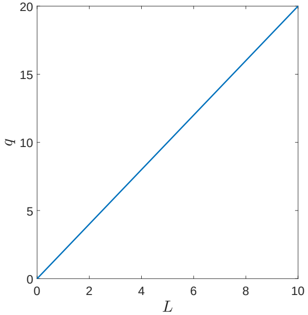

- If $\frac{\partial^2 f}{\partial L^2} = \frac{\partial MP_L}{\partial L} < 0$, then $MP_L$ is diminishing
  - $q$ is concave in $L$
- Each additional unit of labor adds less than the previous unit

# Increasing Marginal Product of Labor




- If $\frac{\partial^2 f}{\partial L^2} = \frac{\partial MP_L}{\partial L} > 0$, then $MP_L$ is increasing
  - $q$ is convex in $L$
- Each additional unit of labor adds more than the previous unit

# Constant Marginal Product of Labor



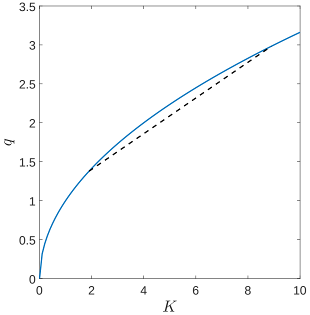

- If $\frac{\partial^2 f}{\partial L^2} = \frac{\partial MP_L}{\partial L} = 0$, then $MP_L$ is constant
  - $q$ is linear in $L$
- Each additional unit of labor adds the same as the previous unit

# Marginal Product of Capital

$$MP_K = \frac{\partial f}{\partial K}$$

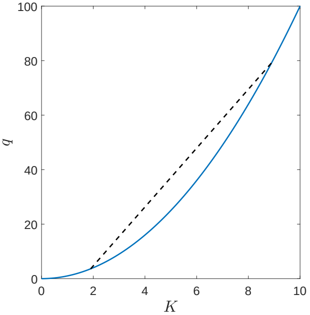
- Similarly, the *marginal product of capital*, denoted by $MP_K$, is the change in production $q$ induced by adding one unit of capital
- We typically assume $MP_K \geq 0$
- Huge surprise, $MP_K$ can be increasing, decreasing, or constant
  - Which case we fall in depends on the sign of $\frac{\partial^2 f}{\partial K^2} = \frac{\partial MP_K}{\partial MP_K}$
- Let's quickly go through each of the three cases

# Diminishing Marginal Product of Capital



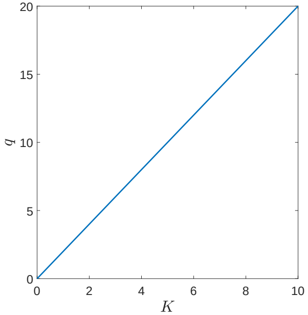

- If $\frac{\partial^2 f}{\partial K^2} = \frac{\partial MP_K}{\partial K} < 0$, then $MP_K$ is diminishing
  - $q$ is concave in $K$
- Each additional unit of capital adds less than the previous unit

# Increasing Marginal Product of Capital




- If $\frac{\partial^2 f}{\partial K^2} = \frac{\partial MP_K}{\partial K} > 0$, then $MP_K$ is increasing
  - $q$ is convex in $K$
- Each additional unit of capital adds more than the previous unit

# Constant Marginal Product of Capital

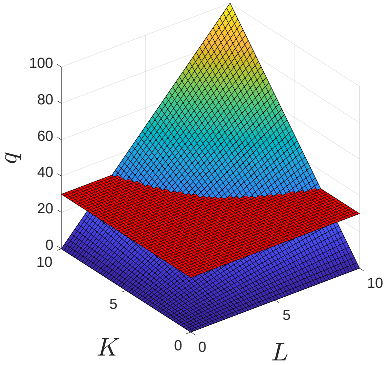



- If $\frac{\partial^2 f}{\partial K^2} = \frac{\partial MP_K}{\partial K} = 0$, then $MP_K$ is constant
  - $q$ is linear in $K$
- Each additional unit of capital adds the same as the previous unit

# Isoquants

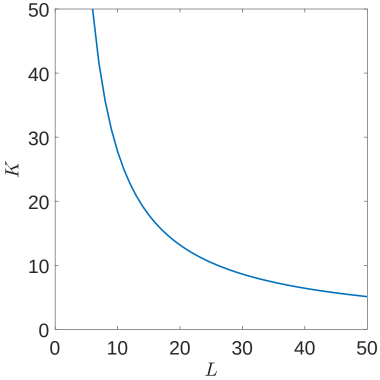
- We looked at plotting $q$ as $L$ changes, holding $K$ fixed
- We looked at plotting $q$ as $K$ changes, holding $L$ fixed
- Lastly, we can plot how $K$ changes with $L$, holding $q$ fixed
- Such a curve is called an *isoquant*, and contains all combinations $(L, K)$ which yield the same output $q$

# Isoquants



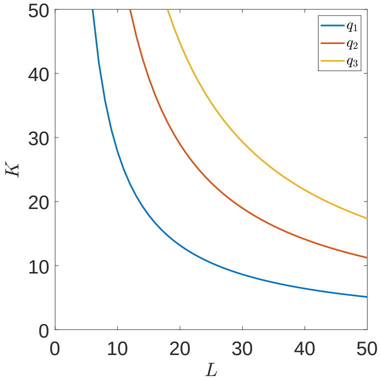

- If we fix quantity $q$ at some level, we see a curve form
- This is a *level curve* of $q(K, L)$

# Isoquants



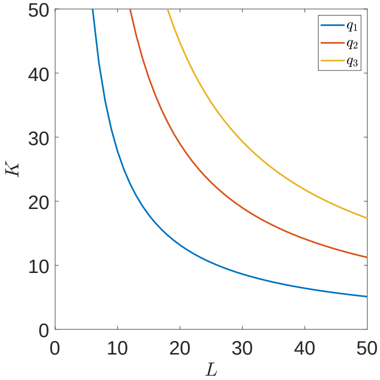

- Each isoquant is associated with a given level of $q$
- Essentially the same thing as an indifference curve

# Isoquants




- Again, we assume $MP_L \geq 0$ and $MP_K \geq 0$
- In this case, quantity increases as we move towards the top right
  - $q_3 > q_2 > q_1$

# Slope of Isoquants




- The slope of an isoquant is given by:

$$slope(isoquant) = -\frac{MP_L}{MP_K} = -MRTS$$

# Marginal Rate of Technical Substitution

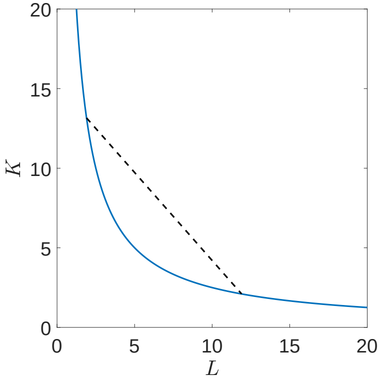
$$MRTS = \frac{MP_L}{MP_K} = \frac{\frac{\partial f}{\partial L}}{\frac{\partial f}{\partial K}} = \frac{\partial K}{\partial L}$$

- The *marginal rate of technical substitution*, or $MRTS$, gives the amount of $K$ the firm could substitute for one additional unit of $L$ such that $q$ is unchanged
- Ex: if $MRTS = 2$ this states that the firm can maintain the same output by sacrificing 2 $K$ and adding 1 $L$
- It's the same thing as the $MRS$
    - The "technical" comes from the fact that $L$ and $K$ are referred to as "production technologies"

# Curvature of Isoquants

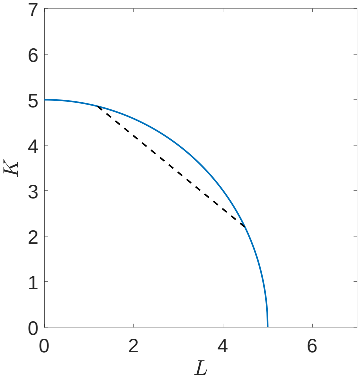
- Just like we saw in consumer theory, the curvature of a production function's isoquants will have major implications for the firm's decision problem
- Isoquants can be:
  - Convex
  - Concave
  - Linear
- Which of the three cases we fall into depends on how the $MRTS$ changes along the isoquants
  - i.e. it depends on the sign of $\frac{\partial MRTS}{\partial L}$

# Convex Isoquant



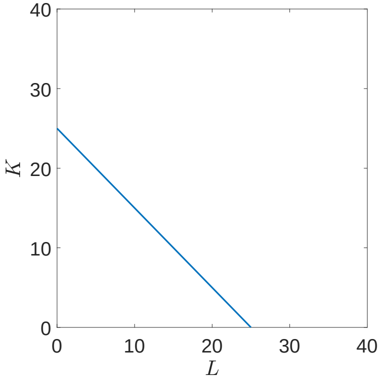

- If $\frac{\partial MRTS}{\partial L} < 0$, isoquant gets flatter as $L$ increases
- Isoquant is convex

# Concave Isoquant




- If $\frac{\partial MRTS}{\partial L} > 0$, isoquant gets steeper as $L$ increases
- Isoquant is concave

# Linear Isoquant




- If $\frac{\partial MRTS}{\partial L} = 0$, isoquant has the same slope as $L$ increases
- Isoquant is linear

# Taking Inventory

$$q = f(K, L)$$

- Given a production function, there are a number of objects of interest
- The $MP_L$ describes how $L$ impacts $q$
- The $MP_K$ describes how $K$ impacts $q$
- The $MRTS$ describes how $K$ can be substituted for $L$ such that $q$ is unchanged
- As you may realize by now, the math underlying consumer and firm theory is essentially the same

# Returns to Scale

- We've used marginal products to think about the impact on $q$ of increasing $K$ or $L$ one at a time
- How about scaling both up at the same time?
- In particular, suppose we scale both $K$ and $L$ up by $t > 1$:

$$f(tL, tK) = ?$$

- For example, if $t = 2$, then we double both labor and capital
- What happens to output when we do this?
- Here, we'll introduce a concept referred to as *returns to scale*

# Increasing Returns to Scale

$$f(tL, tK) > tq$$

- If the above holds, then $f(L, K)$ exhibits *increasing returns to scale*
- Scaling up inputs yields more than directly scaling up output
  - Ex: doubling inputs yields more than double output
- Output "explodes" upwards as $L$ and $K$ are scaled up
- It is increasingly beneficial to scale up the size of the firm

# Decreasing Returns to Scale

$$f(tL, tK) < tq$$

- If the above holds, then $f(L, K)$ exhibits *decreasing returns to scale*
- Scaling up inputs yields less than directly scaling up output
  - Ex: doubling inputs yields less than double output
- Output "levels off' as $L$ and $K$ are scaled up
- It is of limited benefit to scale up the size of the firm

# Constant Returns to Scale

$$f(tL, tK) = tq$$

- If the above holds, then $f(L, K)$ exhibits *constant returns to scale*
- Scaling up inputs yields the same as directly scaling up output
  - Ex: doubling inputs yields double output
- Constant return to scaling up $L$ and $K$
- The benefits of upscaling the size of the firm are always the same

# Returns to Scale

- Given some production function $f(L, K)$, how do we determine whether it exhibits IRS, DRS, or CRS?
- In principle, simple, just compare the two quantities:

$$tq \quad \text{vs} \quad f(tL, fK)$$

- Let's go through an example

# Returns to Scale (Example #1)

$$q = L^{1/2}K^{1/2}$$

- Let's suppose we're given the *Cobb-Douglas* production function above
- Comparing $tq$ versus $f(tL, tK)$:

$$\begin{aligned} tq &\quad \text{vs} \quad (tL)^{1/2}(tK)^{1/2} \\ tq &\quad \text{vs} \quad t^{1/2}L^{1/2}t^{1/2}K^{1/2} \\ tq &\quad \text{vs} \quad tL^{1/2}K^{1/2} \\ tq &= tq \end{aligned}$$

- Here, $tq = f(tL, tK)$, so this production function exhibits constant returns to scale

# Returns to Scale (Example #2)

$$q = L^2 K^2$$

- Suppose instead we're given the production function above
- Comparing $tq$ versus $f(tL, tK)$:

$$tq \quad \text{vs} \quad (tL)^2(tK)^2$$
$$tq \quad \text{vs} \quad t^2L^2t^2K^2$$
$$tq \quad \text{vs} \quad t^4L^2K^2$$
$$tq < t^4q$$

- Since $tq < f(tL, tK)$, this production function exhibits increasing returns to scale

# Returns to Scale (General Cobb-Douglas)

$$q = f(L, K) = cL^a K^b$$

- In general, for Cobb-Douglas:

$$\begin{aligned} f(tL, tK) &= c(tL)^a (tK)^b \\ &= ct^a L^a t^b K^b \\ &= t^{a+b} cL^a K^b \\ &= t^{a+b} q \end{aligned}$$

- Whether we have IRS, DRS, or CRS depends on the magnitude of $a + b$

# Returns to Scale (General Cobb-Douglas)

- If $a + b < 1$, then $tq > f(tL, tK)$ (i.e. DRS) since:

$$tq > t^{a+b}q$$

- If $a + b > 1$, then $tq < f(tL, tK)$ (i.e. IRS) since:

$$tq < t^{a+b}q$$

- If $a + b = 1$, then $tq = f(tL, tK)$ (i.e. CRS) since:

$$tq = t^{a+b}q$$

# Common Production Functions

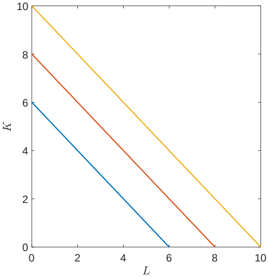
- Let's quickly take a look at some commonly used production functions:
  - Perfect substitutes: $q(L, K) = aL + bK$
  - Perfect complements: $q(L, K) = min\{\frac{L}{a}, \frac{K}{b}\}$
  - Cobb Douglas: $q(L, K) = cL^a K^b$

# Perfect Substitutes

$$q(L, K) = aL + bK$$

- Perfect substitute production functions are used when $L$ and $K$ are seen as perfectly substitutable
  - i.e. firms only need one or the other to produce, no need to have both
- $q(L, K)$ is linear in both $L$ and $K$
  - An implication: marginal product of both inputs is constant

# Perfect Substitutes



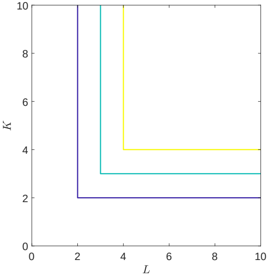

- For perfect substitutes:
$$MRTS = \frac{a}{b}$$
- $\frac{\partial MRTS}{\partial L} = 0$, so slope of isoquant does not change as we move from left to right
- Isoquants have constant slope as a result

# Fixed Proportion (or Perfect Complements)

$$q(L, K) = \min\{\frac{L}{a}, \frac{K}{b}\}$$

- Perfect complements (or fixed proportion) production functions are used when $L$ and $K$ must be used in **fixed proportion** to produce. For example:
- Notice that if $L = 0$ or $K = 0$, $q(L, K) = 0$
  - Need a positive amount of both inputs to produce anything
- **Interpreting** $a$ **and** $b$: the firm needs $a$ units of $L$ and $b$ units of $K$ per unit of output $q$

# Fixed Proportion (or Perfect Complements)



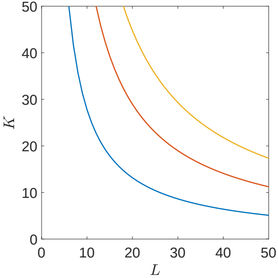

- Perfect complements production functions have L-shaped isoquants
- **Interpretation:** production increases only when firms increase $L$ and $K$ together, not one or the other

# Cobb-Douglas

$$q(L, K) = cL^a K^b$$

- As we saw a moment ago, above is the general form of the Cobb-Douglas production function
- Just like in the utility section: Cobb-Douglas production functions always have convex isoquants

# Cobb Douglas




- Cobb-Douglas isoquants are always strictly convex
- As a result, we'll always have interior solutions when finding optimal production bundles $(L^*, K^*)$