Utility Maximization

ECON 410

May 23, 2023

Utility Maximization

- A DM has preferences described by $u(x, y)$
  - Generally, they like x ($MU_x > 0$) and they like y ($MU_y > 0$)
- But they face a budget constraint: $I \geq p_x x + p_y y$
- How should they allocate their income between x and y so that they maximize their utility?
- Examples:
  - Optimal combination of food and drinks
  - Optimal combination spending and saving
- In this chapter, we will focus on how to solve problems like this

Utility Maximization

- The two ingredients for a utility maximization problem are:
  1. Utility function: $u(x, y)$
  2. Budget constraint: $I \geq p_x x + p_y y$
- The utility function describes the DM's preferences
  - What they like and how it affects their well-being
- The budget constraint describes the DM's feasible set of options
  - What they can actually afford
- If we know the DM's preferences and their set of feasible alternatives, we can compute their optimal consumption bundle

Utility Maximization

- Let's characterize a utility maximization problem in its most general form
- Suppose we face standard budget constraint $I \geq p_x x + p_y y$
- We want to maximize a utility function $u(x, y)$ subject to the budget constraint
- To find the optimal bundle (x^*, y^*) which satisfies the budget constraint, we'll use the *Lagrangian* approach

Lagrangian Approach

- The Lagrangian is always of the form:

$$\mathscr{L} = u(x, y) + \lambda(I - p_x x - p_y y)$$

- The utility function plus λ times income minus expenditures
- Recall that λ is the *Lagrange multiplier*
  - Will talk about its interpretation in the next chapter

Lagrangian Approach

$$\mathscr{L} = u(x, y) + \lambda(I - p_x x - p_y y)$$

- To derive the optimal bundle (x^*, y^*), we'll take three *first order conditions*:

$$\frac{\partial \mathscr{L}}{\partial x} = 0$$
$$\frac{\partial \mathscr{L}}{\partial y} = 0$$
$$\frac{\partial \mathscr{L}}{\partial \lambda} = 0$$

- Differentiate the Lagrangian with respect to x, y, and λ then set each partial derivative $= 0$

Lagrangian Approach

$$\mathscr{L} = u(x, y) + \lambda(I - p_x x - p_y y)$$

- Differentiating the Lagrangian shows that the three first order conditions are:

$$\frac{\partial \mathscr{L}}{\partial x} = 0 \iff MU_x - \lambda p_x = 0$$
$$\frac{\partial \mathscr{L}}{\partial y} = 0 \iff MU_y - \lambda p_y = 0$$
$$\frac{\partial \mathscr{L}}{\partial \lambda} = 0 \iff I - p_x x - p_y y = 0$$

- We can use this system of equations to solve for the optimal (x^*, y^*)

Lagrangian Approach

$$
\begin{aligned}
MU_x - \lambda p_x &= 0 \\
MU_y - \lambda p_y &= 0 \\
I - p_x x - p_y y &= 0
\end{aligned}
$$

- Let's look at the last condition first
- The last equation simply states that at the optimal bundle, income equals expenditures:
$$I = p_x x + p_y y$$
- To maximize our utility, we spend all of our money
- Since more is better, we want as much as we can afford

Lagrangian Approach

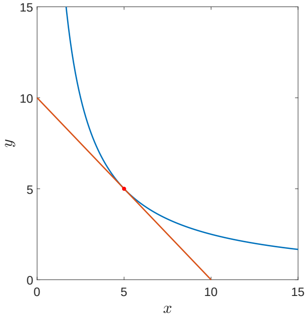
$$MU_x - \lambda p_x = 0$$
$$MU_y - \lambda p_y = 0$$
$$I - p_x x - p_y y = 0$$

- Now let's look at the first two conditions
- Moving prices to the RHS and dividing the first by the second gives:

$$\frac{MU_x}{MU_y} = \frac{p_x}{p_y}$$
$$MRS = MRT$$

- At the optimal bundle, the *MRS* is equal to the *MRT*

Optimal Bundle

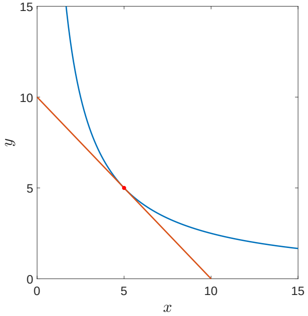


- The three first order conditions show that at the optimal bundle:

$$MRS = MRT$$
$$I = p_x x^* + p_y y^*$$

- $I = p_x x^* + p_y y^* \rightarrow$ optimal bundle lies on budget line
- $MRS = MRT \rightarrow$ slopes of BL and IC are equal

Optimal Bundle



- Recall that when $MU_x > 0$ and $MU_y > 0$, utility increases as we move to the top left
- We want to push the IC as far out as possible, such that it is still touching the budget line

Optimal Bundle

$$\frac{MU_x}{MU_y} = \frac{p_x}{p_y}$$

- The condition above is a tangency condition
  - Equates the slope of the IC and BL
- Another way to understand the condition is by first rewriting it:

$$\frac{MU_x}{p_x} = \frac{MU_y}{p_y}$$

- We refer to $\frac{MU_x}{p_x}$ as the "bang per buck" of x (similar for y)
  - Marginal utility per dollar spent on x
- At the optimal bundle, we equalize the bang per bucks of both goods

Optimal Bundle

- How do we actually solve for the optimal bundle (x^*, y^*)?
- After simplifying our system of first order conditions, we are left with:

$$MRS = MRT$$
$$I = p_x x + p_y y$$

- We have two equations which we can use to solve for the two unknowns (x^*, y^*)
- Let's work through an example

Lagrangian Approach (Example)

- Let's suppose for example:

$$u(x, y) = 2x^{\frac{1}{2}}y^{\frac{1}{2}}$$
$$24 \geq 4x + 2y$$

- First, we write out the Lagrangian:

$$\mathscr{L} = 2x^{\frac{1}{2}}y^{\frac{1}{2}} + \lambda(24 - 4x - 2y)$$

- Next, we take first order conditions

Lagrangian Approach (Example)

- First order conditions:

$$
\begin{aligned}
x^{-\frac{1}{2}} y^{\frac{1}{2}} &= 4\lambda \\
x^{\frac{1}{2}} y^{-\frac{1}{2}} &= 2\lambda \\
24 &= 4x + 2y
\end{aligned}
$$

- Dividing the first by the second gives:

$$\frac{y}{x} = 2$$

Lagrangian Approach (Example)

- We're left with two equations and two unknowns:

$$\frac{y}{x} = 2$$
$$24 = 4x + 2y$$

- Let's use the first equation to isolate y:

$$y = 2x$$

- Then, we can plug this into the budget line:

$$24 = 4x + 2(2x) = 4x + 4x = 8x$$

- Solving for x gives $x^* = 3$

Lagrangian Approach (Example)

- With $x^* = 3$ solved for, we can plug this back into the first equation:

$$\frac{y}{x} = 2$$
$$\frac{y}{3} = 2$$
$$y^* = 6$$

- Then, the optimal bundle is $(x^*, y^*) = (3, 6)$
- In summary:
  1. Write out the Lagrangian
  2. Take 3 first order conditions
  3. Eliminate the Lagrange multiplier λ
  4. Use the remaining two equations to solve for x^* and y^*

Utility Maximization (A Shortcut)

- We just derived the optimal bundle $(3,6)$ given utility function and budget constraint:

$$u(x,y) = 2x^{\frac{1}{2}}y^{\frac{1}{2}}$$
$$24 \geq 4x + 2y$$

- Note that rather than writing out the Lagrangian and taking first order conditions, we could have just started with the conditions:

$$MRS = MRT$$
$$24 = 4x + 2y$$

- Then used these two equations to solve for (x^*, y^*)
- This shortcut works whenever the Lagrangian approach works, but beware...

When does the Lagrangian approach work?

- Think about the Lagrangian as a machine which takes in a utility function and budget line, and tells you where they are tangent
- As long as the optimal bundle (x^*, y^*) is the tangency point between the BL and IC, the Lagrangian will give you the correct answer
- **The optimal bundle is guaranteed to be a tangency point if ICs are convex**
- We call optimal bundles which lie on tangency points "interior solutions"
  - Interior solution: $x > 0$ and $y > 0$ at the optimal bundle

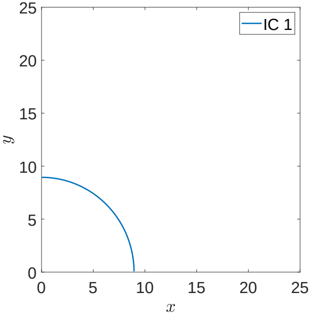
Corner Solutions

- However, not all utility functions have convex ICs
- If the ICs are not convex, the optimal bundle will not be a tangency point
- Instead, we'll have what is called a *corner solutions*
- Corner solutions are optimal bundles which lie on either the x or y axis
    - Corner solution: either $y = 0$ or $x = 0$ at the optimal bundle
- In other words, a corner solution describes a case where it is optimal to either spend all money on x or all money on y

Corner Solutions

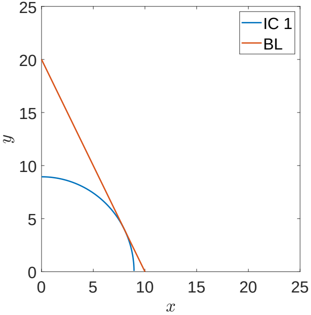
- In summary: the Lagrangian approach spits out the optimal bundle as long as $u(x, y)$ has convex ICs
- If ICs are not convex, Lagrangian will not give the correct answer
- Instead, the optimal bundle will be a corner solution
- Let's go through two cases in which the Lagrangian approach fails:
  1. Concave ICs
  2. Linear ICs

Concave ICs



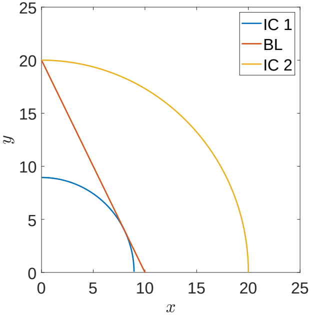
- Suppose that $u(x, y)$ has concave ICs
- Assuming $MU_x > 0$ and $MU_y > 0$, we want to push the IC as far into the top left as possible

Concave ICs



- IC 1 is tangent to the budget line
- This tangency point is the solution that the Lagrangian will give
- However, we can attain a higher level of utility

Concave ICs

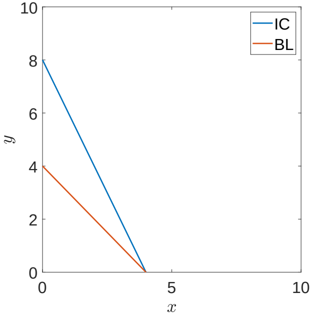


- IC 2 corresponds to a higher level of utility than IC 1, and still touches the budget line
- The optimal bundle in this case is not the tangency point, it is the corner solution $(x, y) = (0, 20)$

Linear ICs

- With linear ICs, there is also always a corner solution
- This is important to remember when dealing with perfect substitute utility
- Whether we are at corner solution $(x, 0)$ or $(0, y)$ depends on the relative slope of the IC and budget line
- Three cases:
  1. $MRS > MRT$ (IC steeper than BL)
  2. $MRS < MRT$ (IC flatter than BL)
  3. $MRS = MRT$ (IC & BL have same slope)

Linear ICs ($MRS > MRT$)

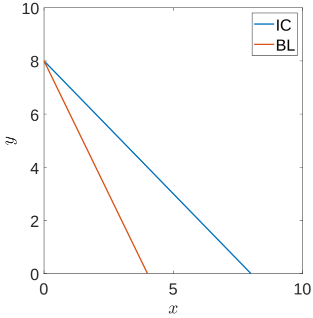


- If $MRS > MRT$, it is optimal to spend all money on x
- Why is this? Notice that:

$$\frac{MU_x}{MU_y} > \frac{p_x}{p_y} \iff \frac{MU_x}{p_x} > \frac{MU_y}{p_y}$$

- If $MRS > MRT$, x has the higher bang per buck

Linear ICs ($MRS < MRT$)



- If $MRS < MRT$, it is optimal to spend all money on y
- Similar to before, notice that:

$$\frac{MU_x}{MU_y} < \frac{p_x}{p_y} \iff \frac{MU_x}{p_x} < \frac{MU_y}{p_y}$$

- If $MRS > MRT$, y has the higher bang per buck

Linear ICs ($MRS = MRT$)

- If ICs are linear, and $MRS = MRT$, then the IC and BL perfectly overlap
  - Slopes are identical
- In this case, any point along the budget line is optimal
- Won't deal with this case too much

Corner Solutions

- We've seen that as long as $u(x, y)$ has convex ICs, the Lagrangian approach will yield the correct answer
- If ICs are not convex, we'll have a corner solution
- There are two possible corner solutions:

$$(x^*, y^*) = \left(\frac{I}{p_x}, 0\right) \quad \text{or} \quad (x^*, y^*) = \left(0, \frac{I}{p_y}\right)$$

- How do we determine which alternative is better?

Bang per Buck Approach

- As we've seen, one way is to compare the bang per buck (BPB) for x versus y:

$$\frac{MU_x}{p_x} \quad \text{vs} \quad \frac{MU_y}{p_y}$$

- It will always be optimal to spend all money on the good with the higher BPB
- With convex ICs, the two BPBs will be equal at the optimal bundle
- With non-convex ICs, the two quantities will not be equal at the optimal bundle

Bang per Buck Approach (Example)

$$u(x, y) = 4x + 3y$$
$$I = 40, \ p_x = 2, \ \ p_y = 1$$

- $u(x, y)$ is a perfect substitutes utility function, so it will have linear ICs
    - Since they are not strictly convex, we will have a corner solution
- To find the optimal bundle, we compare $\frac{MU_x}{p_x}$ versus $\frac{MU_y}{p_y}$
- Doing so yields:
$$\frac{MU_x}{p_x} = 2 < 3 = \frac{MU_y}{p_y}$$
- BPB is higher for y, so we should only purchase y

Bang per Buck Approach (Example)

$$u(x, y) = 4x + 3y$$
$$I = 40, \; p_x = 2, \; p_y = 1$$

- We've concluded that it is optimal to spend all money on y
- To figure out how much we actually purchase, use the budget constraint:

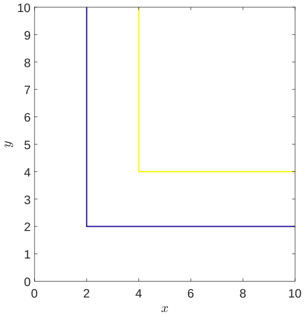
$$40 = 2x + y$$
$$40 = 2(0) + y$$
$$40 = y$$

- The optimal bundle is thus $(x^*, y^*) = (0, 40)$

Special Cases

- Generally, to solve a utility maximization problem, take the following steps:
  1. Verify whether the given $u(x, y)$ has convex ICs $(dMRS/dx < 0)$
  2. If so, set $MRS = MRT$ and use this equation along with the BL to solve for (x^*, y^*)
  3. If not, corner solution $\rightarrow$ compare $\frac{MU_x}{p_x}$ vs $\frac{MU_y}{p_y}$
- There are two other cases which deserve some attention:
  - Perfect complements utility
  - Quasi-linear utility

Perfect Complements

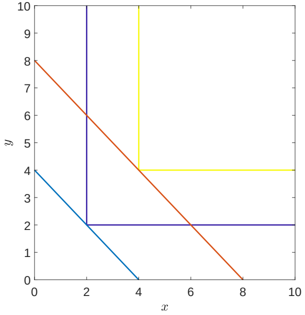


- Recall the perfect complements utility function:

$$u(x, y) = \min\{ax, by\}$$

- Perfect complements utility has L-shaped ICs

Perfect Complements



- With perfect complements, optimal bundles always lie on the "kink points"
- How do we find these bundles?

Perfect Complements

$$u(x, y) = \min\{ax, by\}$$

- Problem: this utility function is not differentiable
  - If we can't take derivatives, how can we possibly derive the MRS?
- Instead, we take a different approach
- Rather than setting $MRS = MRT$ (not possible), we set the two things inside the "min" equal to each other:

$$ax = by$$

Perfect Complements

$$ax = by$$

- Then, we have two equations and 2 unknowns which we can use to solve for x^* and y^*
- Similar in flavor to the standard case
- But, rather than setting $MRS = MRT$, we have the equation $ax = by$
- Let's work through an example

Perfect Complements

$$u(x, y) = min\{3x, 6y\}$$
$$24 = x + 2y$$

- Begin by setting equal the things in the min:

$$3x = 6y$$
$$x = 2y$$

- Plug into the budget line:

$$24 = 2y + 2y$$
$$6 = y^*$$

Perfect Complements

- We've determined that $y^* = 6$. To get x^*, simply plug y^* into our original equation:

$$x = 2y$$
$$x = 2(6)$$
$$x^* = 12$$

- The optimal bundle is thus $(x^*, y^*) = (12, 6)$
- Easy

Quasi-Linear Utility

- Recall the quasi-linear (QL) utility function:

$$u(x, y) = f(x) + by$$

- QL utility functions can yield either corner solutions or interior solutions, so we should handle these with care

- To see what can go wrong, let's work through an example:

$$u(x, y) = 2y + \sqrt{x}$$
$$2 = x + 8y$$

Quasi-Linear Utility

- First, set $MRS = MRT$:

$$\frac{x^{-1/2}}{4} = \frac{1}{8}$$
$$x^* = 4$$

- Plug x^* into the BL:

$$2 = 4 + 8y$$
$$-\frac{1}{4} = y^*$$

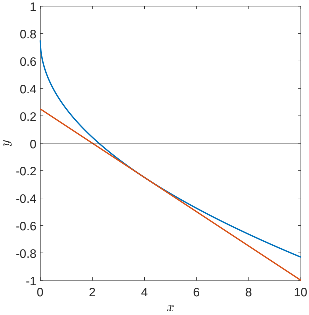
- It seems that the optimal bundle is $(x^*, y^*) = (4, -\frac{1}{4})$
- What happened?

Quasi-Linear Utility

$$MRS = MRT$$
$$I = p_x x + p_y y$$

- Remember that the equations above, which we used to solve for x and y, describe the tangency point between BL and IC
- It is possible, and happens occasionally with QL utility, that the tangency point is outside the first quadrant of the xy plane

Quasi-Linear Utility



- The tangency point $(4, -\frac{1}{4})$ lies outside the first quadrant
- This point is of course not feasible
- How should we proceed in this case?

Quasi-Linear Utility

- Given a quasi-linear utility function, if the tangency point involves negative values of x or y, this is a hint that we have a corner solution
- Intuition: can't consume negative amount of a good, best we can do is consume 0 of it
- How to proceed: spend no money on the good that came out negative, spend all money on good that came out positive

Quasi-Linear Utility

$$u(x, y) = 2y + \sqrt{x}$$
$$2 = x + 8y$$

- We have corner solution $(x^*, y^*) = (\frac{I}{p_x}, 0)$
- The values of p_x and I imply that the optimal bundle is thus: $(x^*, y^*) = (2, 0)$
- This example was tricky. Let's modify and rework it.

Quasi-Linear Utility

$$u(x, y) = 2y + \sqrt{x}$$
$$2 = x + 2y$$

- Here, I've just changed p_y from 8 to 2
- First, set $MRS = MRT$:

$$\frac{x^{-1/2}}{4} = \frac{1}{2}$$
$$x^* = \frac{1}{4}$$

Quasi-Linear Utility

- Plug $x^* = \frac{1}{4}$ into the budget line:

$$\begin{aligned} 2 &= x + 2y \\ 2 &= \frac{1}{4} + 2y \\ \frac{7}{8} &= y^* \end{aligned}$$

- The optimal bundle is thus $(x^*, y^*) = (\frac{1}{4}, \frac{7}{8})$
- This time, we got an interior solution
- No further work to be done at this point

Quasi-Linear Utility

- In summary:
  - QL utility sometimes yields corner solutions
  - To derive the optimal bundle, set $MRS = MRT$ and proceed as usual
  - If $x^* \geq 0$ and $y^* \geq 0 \rightarrow$ interior solution, all done
  - If $x^* < 0$ or $y^* < 0$, we know we have a corner solution
  - If we have a corner solution, spend all money on "non-linear" good, consume 0 of the other good